JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **203,** 187–210 (1996)
ARTICLE NO. 0374

A Reduction Theorem for Systems of Differential Equations with Impulse Effect in a Banach Space

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Received November 18, 1992

A reduction theorem for systems of differential equations with impulse effect at fixed moments in a Banach space is proven. This result allows one to substantially reduce the given system to a much simpler one.

1. INTRODUCTION

Differential equations with impulses provide an adequate mathematical model of evolutionary processes that suddenly change their state at certain moments. The first investigators of differential equations with impulses were A. D. Myshkis and V. D. Mil'man [1]. Later on, monographs on the subject by V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov [2] and A. M. Samoilenko and N. A. Perestyuk [3] were published.

The equivalence problem in the theory of ordinary differential equations was explored by D. M. Grobman [4], P. Hartman [5], and other mathematicians [6–19]. This problem involving the impulse effect was first considered by the author and L. Sermone in [20–23] and D. D. Bainov, S. I. Kostadinov, and Nguyen Van Minh in [24, 25]. In the present paper, a reduction theorem for systems of differential equations with impulses in a Banach space is proven assuming that the system splits into two parts.

*This work has been performed with the financial support of Latvian Council of Science under Grants 90. 224 and 93. 809. E-mail address: reinf@cclu.lv.

0022-247X/96 $18.00

2. STATEMENT OF THE THEOREMS

Let $\mathbf{X}$ and $\mathbf{Y}$ be complex Banach spaces, and $\mathscr{L}(\mathbf{X})$ and $\mathscr{L}(\mathbf{Y})$ be the Banach spaces of linear bounded operators. Consider the following system of differential equations with impulse effect at fixed moments,

$$\begin{cases} dx/dt = A(t)x + f(t,x,y), \\ dy/dt = B(t)y + g(t,x,y), \\ \Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) \\ \qquad\qquad = D_i x(\tau_i - 0) + p_i(x(\tau_i - 0), y(\tau_i - 0)), \\ \Delta y|_{t=\tau_i} = y(\tau_i + 0) - y(\tau_i - 0) \\ \qquad\qquad = E_i y(\tau_i - 0) + q_i(x(\tau_i - 0), y(\tau_i - 0)) \end{cases} \tag{1}$$

and its linear truncation

$$\begin{cases} dx/dt = A(t)x, \\ dy/dt = B(t)y, \\ \Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = D_i x(\tau_i - 0), \\ \Delta y|_{t=\tau_i} = y(\tau_i + 0) - y(\tau_i - 0) = E_i y(\tau_i - 0), \end{cases} \tag{2}$$

where:

(i) the maps $A\colon \mathbf{R} \to \mathscr{L}(\mathbf{X})$ and $B\colon \mathbf{R} \to \mathscr{L}(\mathbf{Y})$ are locally integrable in the Bochner sense;

(ii) the maps $f\colon \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $g\colon \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ are locally integrable in the Bochner sense with respect to t for fixed x and y, and, in addition, they satisfy the estimates

$$|f(t,x,y) - f(t,x',y')| \le \epsilon(|x - x'| + |y - y'|),$$

$$|g(t,x,y) - g(t,x',y')| \le \epsilon(|x - x'| + |y - y'|),$$

$$\sup_{t,x} |g(t,x,0)| < +\infty;$$

(iii) for $i \in \mathbf{Z}$, $D_i \in \mathscr{L}(\mathbf{X})$, $E_i \in \mathscr{L}(\mathbf{Y})$, the maps $p_i\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, $q_i\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ satisfy the estimates

$$|p_i(x,y) - p_i(x',y')| \le \epsilon(|x - x'| + |y - y'|),$$

$$|q_i(x,y) - q_i(x',y')| \le \epsilon(|x - x'| + |y - y'|),$$

$$\sup_{i,x} |q_i(x,0)| < +\infty;$$

(iv) the maps $(x, y) \mapsto (x + D_i x + p_i(x, y), y + E_i y + q_i(x, y))$, $(x, y) \mapsto (x + D_i x, y + E_i y)$ are homeomorphisms;

(v) the moments τ_i of impulse effect form a strictly increasing sequence

$$\cdots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots,$$

where the limit points may be only $\pm\infty$.

DEFINITION 1. By a *solution* to a system of differential equations with impulse effect at fixed moments we mean a piecewise absolutely continuous map with discontinuities of the first kind at the points $t = \tau_i$, which for almost all t satisfies system (1) and for $t = \tau_i$ satisfies the conditions of a "jump."

Note that condition (iv) implies continuability of solutions to (1) and (2) in the negative direction. Furthermore, condition (v) together with the Lipschitz property with respect to x and y of the right-hand side ensures that there is a unique solution defined on $\mathbf{R}$.

Let $U(t, \tau)$ and $V(t, \tau)$ be Cauchy evolution operators of the system of linear equations with impulse effect at fixed moments (2). We assume that the evolution operators satisfy the estimates

$$\nu = \max\Bigg(\sup_t \int_{-\infty}^{t} |V(t,\tau)|\,|U(\tau,t)|\,d\tau + \sup_t \sum_{\tau_i \le t} |V(t,\tau_i)|\,|U(\tau_i - 0, t)|,$$

$$\sup_t \int_t^{+\infty} |V(\tau,t)|\,|U(t,\tau)|\,d\tau + \sup_t \sum_{t<\tau_i} |V(\tau_i - 0, t)|\,|U(t,\tau_i)| \Bigg) < +\infty,$$

$$\mu = \sup_t \left(\int_{-\infty}^{t} |V(t,\tau)|\,d\tau + \sum_{\tau_i \le t} |V(t,\tau_i)| \right) < +\infty.$$

It should be noted that if the linear system (2) is autonomous and has no impulse effect, then the Cauchy operators are $U(t, \tau) = e^{A(t-\tau)}$ and $V(t, \tau) = e^{B(t-\tau)}$. Therefore $\mu = \int_0^{+\infty} |e^{B\tau}|\,d\tau$ and $\nu = \int_0^{+\infty} |e^{-A\tau}|\,|e^{B\tau}|\,d\tau$. Consequently, the integrals converge if the spectrum of the map B is located to the left of the spectrum of the map A and the spectra are separated by a vertical line in the left complex half-plane.

Let $\Phi(\cdot, t_0, x_0, y_0) = (x(\cdot, t_0, x_0, y_0), y(\cdot, t_0, x_0, y_0))\colon \mathbf{R} \to \mathbf{X} \times \mathbf{Y}$ be the solution of system (1), where $\Phi(t_0 + 0, t_0, x_0, y_0) = (x_0, y_0)$. At the break points τ_i the values for all solutions are taken at $\tau_i + 0$ unless otherwise specified. For short, we will use the notation $\Phi(t) = (x(t), y(t))$.

THEOREM 1. *Let* $2\epsilon \max\{\sup_i |(\mathrm{id}_x + D_i)^{-1}|, \mu\} < 1 + \sqrt{1 - 4\epsilon\nu}$ *and* $4\nu\epsilon \leq 1$. *Then there exists a unique piecewise continuous bounded map* $G\colon \mathbf{R} \times \mathbf{X} \to \mathbf{Y}$ *with the following properties*:

(i) $G(t, x(t, t_0, x_0, G(t_0, x_0))) = y(t, t_0, x_0, G(t_0, x_0))$ *for all* $t \in \mathbf{R}$;

(ii) $|G(t, x_0) - G(t, x_0')| \leq \lambda |x_0 - x_0'|$;

(iii) $\int_{t_0}^{+\infty} |U(t_0, t)|\, |y(t, t_0, x_0, y_0) - G(t, x(t, t_0, x_0, y_0))|\, dt + \sum_{t_0 < \tau_i} |U(t_0, \tau_i)|\, |y(\tau_i - 0, t_0, x_0, y_0) - G(\tau_i - 0, x(\tau_i - 0, t_0, x_0, y_0))| \leq \nu(1 - \epsilon(1 + \lambda)\nu)^{-1} |y_0 - G(t_0, x_0)|$, *where* $\lambda = (2\epsilon\nu)^{-1}(1 - 2\epsilon\nu - \sqrt{1 - 4\epsilon\nu})$.

The properties (i), (ii) for impulsive systems with additional assumptions were proven in [26]. The third inequality characterizes the integral distance between an arbitrary solution and the integral manifold. Note that $\lambda < 1$ when $4\nu\epsilon < 1$.

Let $\mathbf{U}$ be a complex Banach space. Consider two systems of differential equations with impulse effect at fixed moments

$$du/dt = P(t, u), \qquad \Delta u|_{t = \tau_i} = S_i\big(u(\tau_i = 0)\big) \tag{3}$$

and

$$du/dt = Q(t, u), \qquad \Delta u|_{t = \tau_i} = T_i\big(u(\tau_i - 0)\big) \tag{4}$$

that satisfy the conditions of the existence and uniqueness theorem. We assume that the maximum interval of the existence of the solutions is $\mathbf{R}$. Let $\phi(\cdot, t_0, u_0)\colon \mathbf{R} \to \mathbf{U}$ and $\psi(\cdot, t_0, u_0)\colon \mathbf{R} \to \mathbf{U}$ be the solutions of the above systems, respectively. Suppose that there is a function $e\colon \mathbf{U} \to \mathbf{R}$ such that

$$\max\Big(|P(t, u) - Q(t, u)|, \sup_i |S_i(u) - T_i(u)|\Big) \leq e(u).$$

DEFINITION 2. The two systems of differential equations (3) and (4) with impulse effect at fixed moments are *globally strongly dynamically equivalent* if there exists a map $H\colon \mathbf{R} \times \mathbf{U} \to \mathbf{U}$ and a positive constant c such that:

(i) $H(t, \cdot)\colon \mathbf{U} \to \mathbf{U}$ is a homeomorphism;

(ii) $H(t, \phi(t, t_0, u_0)) = \psi(t, t_0, H(t_0, u_0))$ for all $t \in \mathbf{R}$;

(iii) $\max(|H(t, u) - u|, |H^{-1}(t, u) - u|) \leq ce(u)$;

(iv) in case the systems of differential equations are autonomous and have no impulse effect, then the map H does not depend on t.

Note that without (iii) and (iv) the concept of dynamical equivalence would be trivial, since in this case the equality $H(t_0, u_0) = \psi(t_0, 0, \phi(0, t_0, u_0))$ gives a dynamical equivalence. It is significant that in the case of the classical global Grobman–Hartman theorem [4, 5] for autonomous differential equations, the corresponding function $e(x) = a > 0$ and appropriate constant c depend on linear truncation only.

Next, consider a reduced system of differential equations with impulse effect at fixed moments

$$\begin{cases} dx/dt = A(t)x + f(t, x, G(t, x)), \\ dy/dt = B(t)y + g(t, 0, y), \\ \Delta x|_{t=\tau_i} = D_i x(\tau_i - 0) + p_i\big(x(\tau_i - 0), G(\tau_i - 0, x(\tau_i - 0))\big), \\ \Delta y|_{t=\tau_i} = E_i y(\tau_i - 0) + q_i(0, y(\tau_i - 0)). \end{cases} \tag{5}$$

The last system splits into two parts. The first of them does not contain the variable y, while the second one is independent of x. In case

$$\epsilon \max\left\{2\left(1 + \sqrt{1 - 4\epsilon\nu}\right)^{-1} \sup_i \left|\left(\mathrm{id}_x + D_i\right)^{-1}\right|, \sup_i \left|\left(\mathrm{id}_y + E_i\right)^{-1}\right|\right\} < 1,$$

the maps $(x, y) \mapsto (x + D_i x + p_i(x, G(\tau_i - 0, x)), y + E_i y + q_i(0, y))$ are homeomorphisms. This condition ensures continuability of solutions to (5) on $\mathbf{R}$.

Let $\Psi(\cdot, t_0, x_0, y_0) = (x_0(\cdot, t_0, x_0), y_0(\cdot, t_0, y_0)): \mathbf{R} \to \mathbf{X} \times \mathbf{Y}$ be a solution of system (5), where $\Psi(t_0 + 0, t_0, x_0, y_0) = (x_0, y_0)$. For short, we will use the notation $\Psi(t) = (x_0(t), y_0(t))$.

From here on we assume that

$$\sup_{t, x, y} |g(t, x, y) - g(t, 0, y)| < +\infty, \tag{6}$$

$$\sup_{i, x, y} |q_i(x, y) - q_i(0, y)| < +\infty. \tag{7}$$

THEOREM 2. *Let* $4\nu\epsilon < 1$, $2\epsilon\mu < 1 + \sqrt{1 - 4\epsilon\nu}$ *and* $\epsilon \max\{2(1 + \sqrt{1 - 4\epsilon\nu})^{-1} \sup_i |(\mathrm{id}_x + D_i)^{-1}|, \sup_i |(\mathrm{id}_y + E_i)^{-1}|\} < 1$. *Then systems* (1) *and* (5) *are globally strongly dynamically equivalent*.

Remark. If instead of (6) and (7) we assume that $\sup_{t,x,y} |g(t, x, y)| < +\infty$ and $\sup_{i,x,y} |q_i(x, y)| < +\infty$, then systems (1) and

$$\begin{cases} dx/dt = A(t)x + f(t, x, G(t, x)), \\ dy/dt = B(t)y, \\ \Delta x|_{t=\tau_i} = D_i x(\tau_i - 0) + p_i\big(x(\tau_i - 0), G(\tau_i - 0, x(\tau_i - 0))\big), \\ \Delta y|_{t=\tau_i} = E_i y(\tau_i - 0) \end{cases}$$

are globally strongly dynamically equivalent.

3. AUXILIARY LEMMA

Let $\mathbf{PC}(\mathbf{R} \times \mathbf{X}, \mathbf{Y})$ be a set of maps $G: \mathbf{R} \times \mathbf{X} \to \mathbf{Y}$ that are continuous for $(t, x) \in [\tau_i, \tau_{i+1}) \times \mathbf{X}$ and have discontinuities of the first kind for $t = \tau_i$. The set

$$\mathscr{S}_0 = \left(G \in \mathbf{PC}(\mathbf{R} \times \mathbf{X}, \mathbf{Y}) \middle| \sup_{t, x} |G(t, x)| < +\infty\right)$$

becomes a Banach space if we use the norm $\|G\| = \sup_{t, x} |G(t, x)|$. It is obvious that $\mathscr{S}_0$ is a linear normed space. It remains to prove that $\mathscr{S}_0$ is a complete space. So $\mathbf{Y}$ is a Banach space. In the usual way we prove that the restriction of the limit map on $[\tau_i, \tau_{i+1}) \times \mathbf{X}$ is continuous and $\lim_{t \to \tau_{i+1} - 0} G(t, x)$ exists. For $\lambda > 0$ the set

$$\mathscr{M}(\lambda) = (G \in \mathscr{S}_0 | \, |G(t, x) - G(t, x')| \leq \lambda |x - x'|)$$

is a closed subset of $\mathscr{S}_0$.

Let $\zeta: \mathbf{R} \to \mathbf{X}$ be a solution of the following equation with impulses:

$$\begin{cases} d\zeta/dt = A(t)\zeta + f(t, \zeta, G(t, \zeta)), \zeta(t_0) = x_0, \\ \Delta\zeta|_{t = \tau_i} = D_i \zeta(\tau_i - 0) + p_i(\zeta(\tau_i - 0), G(\tau_i - 0, \zeta(\tau_i - 0))). \end{cases} \tag{8}$$

Since $\epsilon(1 + \lambda)\sup_i |(\mathrm{id}_x + D_i)^{-1}| = 2\epsilon(1 + \sqrt{1 - 4\epsilon\nu})^{-1} \sup_i |(\mathrm{id}_x + D_i)^{-1}| < 1$, the maps $x \mapsto x + D_i x + p_i(x, G(\tau_i - 0, x))$ are homeomorphisms. So the solutions ζ of (8) are defined on $\mathbf{R}$, and for $t \geq t_0$ they can be represented in the form

$$\zeta(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \tau) f(\tau, \zeta(\tau), G(\tau, \zeta(\tau)))\, d\tau + \sum_{t_0 < \tau_i \leq t} U(t, \tau_i) p_i(\zeta(\tau_i - 0), G(\tau_i - 0, \zeta(\tau_i - 0))),$$

while for $t \leq t_0$ their representation will be

$$\zeta(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \tau) f(\tau, \zeta(\tau), G(\tau, \zeta(\tau)))\, d\tau - \sum_{t < \tau_i \leq t_0} U(t, \tau_i) p_i(\zeta(\tau_i - 0), G(\tau_i - 0, \zeta(\tau_i - 0))).$$

Here, for short, the notation $\zeta(t) = \zeta(t, t_0, x_0)$ is used.

In the proof of the main theorem we will use the following lemma.

LEMMA 1. *Let $G, G' \in \mathscr{M}(\lambda)$ and let $\epsilon(1+\lambda)\nu < 1$. Then the following estimate is valid*:

$$\int_{-\infty}^{t_0} |V(t_0,t)|\,|\zeta(t)-\zeta'(t)|\,dt + \sum_{\tau_i \le t_0} |V(t_0,\tau_i)|\,|\zeta(\tau_i-0)-\zeta'(\tau_i-0)| \le \nu\big(1-\epsilon(1+\lambda)\nu\big)^{-1}\big(|x_0-x_0'| + \epsilon\mu\|G-G'\|\big).$$

Proof. Using the estimates on f and p_i we get for $t \le t_0$

$$\begin{aligned} |\zeta(t)-\zeta'(t)| &\le |U(t,t_0)|\,|x_0-x_0'| + \epsilon(1+\lambda) \\ &\quad \times\Bigg(\int_t^{t_0} |U(t,\tau)|\,|\zeta(\tau)-\zeta'(\tau)|\,d\tau \\ &\quad + \sum_{t<\tau_i\le t_0} |U(t,\tau_i)|\,|\zeta(\tau_i-0)-\zeta'(\tau_i-0)|\Bigg) \\ &\quad + \epsilon\|G-G'\|\Bigg(\int_t^{t_0}|U(t,\tau)|\,d\tau + \sum_{t<\tau_i\le t_0}|U(t,\tau_i)|\Bigg). \end{aligned} \tag{9}$$

Multiplying (9) by $V(t_0,t)$, integrating from $-\infty$ to t_0, and changing the order of integration, we obtain

$$\begin{aligned} &\int_{-\infty}^{t_0} |V(t_0,t)|\,|\zeta(t)-\zeta'(t)|\,dt \\ &\quad \le \sup_{\tau}\int_{-\infty}^{\tau} |V(\tau,t)|\,|U(t,\tau)|\,dt \\ &\qquad \times\Bigg(|x_0-x_0'| + \epsilon(1+\lambda)\Bigg(\int_{-\infty}^{t_0}|V(t_0,\tau)|\,|\zeta(\tau)-\zeta'(\tau)|\,d\tau \\ &\qquad + \sum_{\tau_i\le t_0}|V(t_0,\tau_i)|\,|\zeta(\tau_i-0)-\zeta'(\tau_i-0)|\Bigg) + \epsilon\mu\|G-G'\|\Bigg). \end{aligned}$$

Multiplying (9) by $V(t_0, \tau_j)$, summing up for all j's with respect to $\tau_j \leq t_0$, and changing the order of summation, we obtain

$$
\begin{aligned}
\sum_{\tau_j \leq t_0} &|V(t_0, \tau_j)|\,|\zeta(\tau_j - 0) - \zeta'(\tau_j - 0)| \\
&\leq \sup_{\tau} \sum_{\tau_j \leq \tau} |V(\tau, \tau_j)|\,|U(\tau_j - 0, \tau)| \\
&\quad \times \Bigg(|x_0 - x_0'| + \epsilon(1 + \lambda)\Bigg(\int_{-\infty}^{t_0} |V(t_0, \tau)|\,|\zeta(\tau) - \zeta'(\tau)|\, d\tau \\
&\qquad + \sum_{\tau_i \leq t_0} |V(t_0, \tau_i)|\,|\zeta(\tau_i - 0) - \zeta'(\tau_i - 0)| \Bigg) + \epsilon\mu \|G - G'\| \Bigg).
\end{aligned}
$$

Denote

$$
\begin{aligned}
\rho = \int_{-\infty}^{t_0} &|V(t_0, t)|\,|\zeta(t) - \zeta'(t)|\, dt \\
&+ \sum_{\tau_i \leq t_0} |V(t_0, \tau_i)|\,|\zeta(\tau_i - 0) - \zeta'(\tau_i - 0)|.
\end{aligned}
$$

From the above inequalities we obtain

$$
\rho \leq \nu\big(1 - \epsilon(1 + \lambda)\nu\big)^{-1}\big(|x_0 - x_0'| + \epsilon\mu\|G - G'\|\big).
$$

And thus, the lemma is proven.

4. PROOF OF THEOREM 1

Proof. We now define an operator $\mathbf{G}\colon \mathscr{M}(\lambda) \to \mathscr{M}(\lambda)$ by the formula

$$
\begin{aligned}
(\mathbf{G}G)(t_0, x_0) = \int_{-\infty}^{t_0} &V(t_0, \tau) g(\tau, \zeta(\tau), G(\tau, \zeta(\tau)))\, d\tau \\
&+ \sum_{\tau_i \leq t_0} V(t_0, \tau_i) q_i\big(\zeta(\tau_i - 0), G(\tau_i - 0, \zeta(\tau_i - 0))\big).
\end{aligned}
$$

If G is a fixed point of operator $\mathbf{G}$, then G defines the integral manifold of system (1). Let $G, G' \in \mathscr{M}(\lambda)$ be arbitrary. The conditions of the theorem imply that $\mathbf{G}G \in \mathscr{S}_0$. Next we get

$$\begin{aligned}
&|(\mathbf{G}G)(t_0, x_0) - (\mathbf{G}G')(t_0, x_0')| \\
&\quad \le \epsilon(1+\lambda)\bigg(\int_{-\infty}^{t_0} |V(t_0,\tau)|\,|\zeta(\tau) - \zeta'(\tau)|\,d\tau \\
&\qquad\qquad + \sum_{\tau_i \le t_0} |V(t_0,\tau_i)|\,|\zeta(\tau_i - 0) - \zeta'(\tau_i - 0)|\bigg) \\
&\qquad + \epsilon\mu\|G - G'\|.
\end{aligned}$$

If $4\nu\epsilon \le 1$, then $\lambda = (2\epsilon\nu)^{-1}(1 - 2\epsilon\nu - \sqrt{1 - 4\epsilon\nu}) > 0$ satisfies the equality

$$\epsilon\nu(1+\lambda)(1 - \epsilon(1+\lambda)\nu)^{-1} = \lambda.$$

From this, $2(\sqrt{1-4\epsilon\nu} + 1)^{-1} = 1 + \lambda$ and $2\epsilon(1+\lambda)\nu \le 1$. Using the result of Lemma 1, we obtain

$$\begin{aligned}
&|(\mathbf{G}G)(t_0, x_0) - (\mathbf{G}G')(t_0, x_0')| \\
&\quad \le \epsilon\nu(1+\lambda)(1 - \epsilon(1+\lambda)\nu)^{-1}|x_0 - x_0'| \\
&\qquad + \epsilon\mu\big(\epsilon\nu(1+\lambda)(1 - \epsilon(1+\lambda)\nu)^{-1} + 1\big)\|G - G'\| \\
&\quad = \lambda|x_0 - x_0'| + \epsilon\mu(1+\lambda)\|G - G'\|.
\end{aligned}$$

If $2\epsilon\mu < 1 + \sqrt{1 - 4\epsilon\nu}$, we get $\epsilon\mu(1+\lambda) < 1$. We obtain that $\mathbf{G}$ is a contraction on $\mathscr{M}(\lambda)$. This implies that in $\mathscr{M}(\lambda)$ there is only one solution satisfying the functional equation of the integral manifold.

It remains to verify that $y = G(t, x)$ is the equation of the integral manifold. It should be noted that $\zeta(t) = \zeta(t, t_0, x_0)$, $\zeta(\tau) = \zeta(\tau, t_0, x_0) = \zeta(\tau, t, \zeta(t, t_0, x_0))$. Let $\eta(t) = G(t, \zeta(t))$ and $t \ge t_0$.

$$\begin{aligned}
\eta(t) &= G(t, \zeta(t)) \\
&= \int_{-\infty}^{t} V(t,\tau)\, g(\tau, \zeta(\tau), G(\tau, \zeta(\tau)))\,d\tau \\
&\quad + \sum_{\tau_i \le t} V(t, \tau_i)\, q_i\big(\zeta(\tau_i - 0), G(\tau_i - 0, \zeta(\tau_i - 0))\big)
\end{aligned}$$

$$= V(t,t_0)\left(\int_{-\infty}^{t_0} V(t_0,\tau)g(\tau,\zeta(\tau),G(\tau,\zeta(\tau)))\,d\tau\right.$$
$$\left. + \sum_{\tau_i\le t_0} V(t_0,\tau_i)q_i\big(\zeta(\tau_i-0),G\big(\tau_i-0,\zeta(\tau_i-0)\big)\big)\right)$$
$$+\int_{t_0}^{t} V(t,\tau)g(\tau,\zeta(\tau),G(\tau,\zeta(\tau)))\,d\tau$$
$$+ \sum_{t_0<\tau_i\le t} V(t,\tau_i)q_i\big(\zeta(\tau_i-0),G\big(\tau_i-0,\zeta(\tau_i-0)\big)\big)$$
$$= V(t,t_0)G(t_0,x_0) + \int_{t_0}^{t} V(t,\tau)g(\tau,\zeta(\tau),\eta(\tau))\,d\tau$$
$$+ \sum_{t_0<\tau_i\le t} V(t,\tau_i)q_i\big(\zeta(\tau_i-0),\eta(\tau_i-0)\big).$$

This means that $(\zeta(\cdot),\eta(\cdot))\colon \mathbf{R}\to\mathbf{X}\times\mathbf{Y}$ is the solution of (1) satisfying the initial conditions $\zeta(t_0)=x_0, \eta(t_0)=G(t_0,x_0)$. The case $t\le t_0$ is treated analogously. From the uniqueness of solutions we get for all $t\in\mathbf{R}$ that

$$G\big(t,x\big(t,t_0,x_0,G(t_0,x_0)\big)\big) = y\big(t,t_0,x_0,G(t_0,x_0)\big).$$

For an arbitrary map $\xi\colon \mathbf{R}\to\mathbf{Y}$, piecewise-continuous from the right with points of discontinuity $t=\tau_i$ of the first type, the following relation holds

$$\xi(t) = \xi(t_0) + \lim_{\delta\to+0}\delta^{-1}\int_{t_0}^{t}(\xi(s+\delta)-\xi(s))\,ds.$$

Set $\xi(s)=V(t,s)G(s,x(s))$. Then for $t\ge t_0$ we obtain

$$G(t,x(t))$$
$$= V(t,t_0)G(t_0,x_0) + \lim_{\delta\to+0}\delta^{-1}$$
$$\times\int_{t_0}^{t}\big(V(t,s+\delta)G(s+\delta,x(s+\delta)) - V(t,s)G(s,x(s))\big)\,ds$$
$$= V(t,t_0)G(t_0,x_0) + \lim_{\delta\to+0}\delta^{-1}\int_{t_0}^{t}V(t,s+\delta)$$
$$\times\big(G(s+\delta,x(s+\delta)) - y(s+\delta,s,x(s),G(s,x(s)))\big)\,ds$$
$$+\lim_{\delta\to+0}\delta^{-1}\int_{t_0}^{t}\big(V(t,s+\delta)y(s+\delta,s,x(s),G(s,x(s)))$$
$$-V(t,s)G(s,x(s))\big)\,ds.$$

Let us note that

$$
\begin{aligned}
y(s+\delta, s, x_1, y_1) = V(s+\delta, s) y_1 &+ \int_s^{s+\delta} V(s+\delta, \tau) g\big(\tau, \Phi(\tau, s, x_1, y_1)\big)\, d\tau \\
&+ \sum_{s<\tau_i\le s+\delta} V(s+\delta, \tau_i) q_i\big(\Phi(\tau_i - 0, s, x_1, y_1)\big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\lim_{\delta\to+0} \delta^{-1} \int_{t_0}^{t} \bigg(\int_s^{s+\delta} V(t,\tau) g\big(\tau, \Phi(\tau, s, x(s), G(s, x(s)))\big)\, d\tau \\
&\qquad + \sum_{s<\tau_i\le s+\delta} V(t,\tau_i) q_i\big(\Phi(\tau_i - 0, s, x(s), G(s, x(s)))\big)\bigg) ds \\
&\quad = \int_{t_0}^{t} V(t,s) g(s, x(s), G(s, x(s)))\, ds \\
&\qquad + \sum_{t_0<\tau_i\le t} V(t,\tau_i) q_i\big(x(\tau_i - 0), G\big(\tau_i - 0, x(\tau_i - 0)\big)\big).
\end{aligned}
$$

Next we obtain

$$
\begin{aligned}
&y(t) - G(t, x(t)) \\
&\quad = V(t,t_0)\big(y_0 - G(t_0, x_0)\big) \\
&\qquad + \int_{t_0}^{t} V(t,s)\big(g(s, x(s), y(s)) - g(s, x(s), G(s, x(s)))\big)\, ds \\
&\qquad + \lim_{\delta\to+0} \delta^{-1} \int_{t_0}^{t} V(t, s+\delta)\big(y(s+\delta, s, x(s), G(s, x(s))) \\
&\qquad\qquad - G(s+\delta, x(s+\delta))\big)\, ds \\
&\qquad + \sum_{t_0<\tau_i\le t} V(t,\tau_i)\big(q_i(x(\tau_i - 0), y(\tau_i - 0)) \\
&\qquad\qquad - q_i\big(x(\tau_i - 0), G\big(\tau_i - 0, x(\tau_i - 0)\big)\big)\big).
\end{aligned}
$$

Now we consider

$$
\begin{aligned}
x(s+\delta, s, x_1, y_1) &- x\big(s+\delta, s, x_1, G(s, x_1)\big) \\
&= \int_s^{s+\delta} U(s+\delta, \tau)\big(f\big(\tau, \Phi(\tau, s, x_1, y_1)\big) \\
&\qquad\qquad -f\big(\tau, \Phi\big(\tau, s, x_1, G(s, x_1)\big)\big)\big)\, d\tau \\
&\quad + \sum_{s<\tau_i\le s+\delta} U(s+\delta, \tau_i)\big(p_i\big(\Phi(\tau_i - 0, s, x_1, y_1)\big) \\
&\qquad\qquad -p_i\big(\Phi\big(\tau_i - 0, s, x_1, G(s, x_1)\big)\big)\big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{\delta\to+0} \delta^{-1}\left|\int_{t_0}^{t} V(t, s+\delta)\big(y\big(s+\delta, s, x(s), G(s, x(s))\big)\right. \\
&\qquad\qquad \left. -G\big(s+\delta, x(s+\delta)\big)\big)\, ds\right| \\
&\quad \le \lambda \lim_{\delta\to+0} \delta^{-1} \int_{t_0}^{t} |V(t, s+\delta)|\,\big|x\big(s+\delta, s, x(s), G(s, x(s))\big) \\
&\qquad\qquad -x\big(s+\delta, x(s), y(s)\big)\big|\, ds \\
&\quad \le \lambda \lim_{\delta\to+0} \delta^{-1} \int_{t_0}^{t} |V(t, s+\delta)| \\
&\qquad \times\left(\int_s^{s+\delta} |U(s+\delta, \tau)|\,\big|f\big(\tau, \Phi(\tau, s, x(s), y(s))\big)\right. \\
&\qquad\quad -f\big(\tau, \Phi\big(\tau, s, x(s), G(s, x(s))\big)\big)\big|\, d\tau \\
&\qquad\quad + \sum_{s<\tau_i\le s+\delta} |U(s+\delta, \tau_i)|\,\big|p_i\big(\Phi(\tau_i - 0, s, x(s), y(s))\big) \\
&\qquad\quad \left. -p_i\big(\Phi\big(\tau_i - 0, s, x(s), G(s, x(s))\big)\big)\big|\right) ds \\
&\quad = \lambda\left(\int_{t_0}^{t} |V(t, s)|\,\big|f\big(s, x(s), y(s)\big) - f\big(s, x(s), G(s, x(s))\big)\big|\, ds\right. \\
&\qquad + \sum_{t_0<\tau_i\le t} |V(t, \tau_i)|\,\big|p_i\big(x(\tau_i - 0), y(\tau_i - 0)\big) \\
&\qquad \left. -p_i\big(x(\tau_i - 0), G(\tau_i - 0, x(\tau_i - 0))\big)\big|\right).
\end{aligned}
$$

Introduce the expression $\xi(t) = |y(t) - G(t, x(t))|$. For $t \geq t_0$ we obtain the estimate

$$\xi(t) \leq |V(t, t_0)| \xi(t_0) + \epsilon(1 + \lambda)\left(\int_{t_0}^{t} |V(t, s)| \xi(s)\, ds + \sum_{t_0 < \tau_i \leq t} |V(t, \tau_i)| \xi(\tau_i - 0)\right).$$

Multiplying by $U(t_0, t)$, integrating and summing analogously as in Lemma 1 we obtain the inequality

$$\int_{t_0}^{+\infty} |U(t_0, t)| \, |y(t, t_0, x_0, y_0) - G(t, x(t, t_0, x_0, y_0))| \, dt + \sum_{t_0 < \tau_i} |U(t_0, \tau_i)| \, |y(\tau_i - 0, t_0, x_0, y_0) - G(\tau_i - 0, x(\tau_i - 0, t_0, x_0, y_0))| \leq \nu(1 - \epsilon(1 + \lambda)\nu)^{-1} |y_0 - G(t_0, x_0)|$$

which proves the theorem.

5. PROOF OF THEOREM 2

The proof of the theorem consists of several steps.

Consider the Banach spaces

$$\mathscr{B}_1 = \left(k \in \mathbf{PC}(\mathbf{R} \times \mathbf{X} \times \mathbf{Y}, \mathbf{X}) \,\middle|\, \sup_{t, x, y} \frac{|k(t, x, y)|}{|y - G(t, x)|} < +\infty\right)$$

and

$$\mathscr{S} = \left(l \in \mathbf{PC}(\mathbf{R} \times \mathbf{X} \times \mathbf{Y}, \mathbf{Y}) \,\middle|\, \sup_{t, x, y} |l(t, x, y)| < +\infty\right)$$

equipped with the norms

$$\|k\| = \sup_{t, x, y} \frac{|k(t, x, y)|}{|y - G(t, x)|}$$

and

$$\|l\| = \sup_{t, x, y} |l(t, x, y)|,$$

respectively. Similar to Lemma 1 we get that $\mathscr{B}_1$ is a linear normed space and there exists a continuous limit map for $(t, x, y) \in [\tau_i, \tau_{i+1}) \times \mathbf{X} \times \mathbf{Y}$, $y \neq G(t, x)$. The continuity of the limit map on the integral manifold follows from the estimate $|k(t, x, y)| \leq \|k\|\, |y - G(t, x)|$.

LEMMA 2. *There exists a unique solution of the functional equations*

$$\begin{aligned} k_1(t_0, x_0, y_0) &= \int_{t_0}^{+\infty} U(t_0, \tau)\big(f(\tau, \Phi(\tau)) \\ &\qquad - f\big(\tau, x(\tau) + k_1(\tau, \Phi(\tau)), G(\tau, x(\tau) + k_1(\tau, \Phi(\tau)))\big)\big)\, d\tau \\ &\quad + \sum_{t_0 < \tau_i} U(t_0, \tau_i)\big(p_i(\Phi(\tau_i - 0)) \\ &\qquad - p_i\big(x(\tau_i - 0) + k_1(\tau_i - 0, \Phi(\tau_i - 0)), \\ &\qquad G\big(\tau_i - 0, x(\tau_i - 0) + k_1(\tau_i - 0, \Phi(\tau_i - 0))\big)\big)\big), \end{aligned}$$

$$\begin{aligned} l_1(t_0, x_0, y_0) &= \int_{-\infty}^{t_0} V(t_0, \tau) \\ &\qquad \times \big(g\big(\tau, 0, y(\tau) + l_1(\tau, \Phi(\tau))\big) - g(\tau, \Phi(\tau))\big)\, d\tau \\ &\qquad + \sum_{\tau_i \leq t_0} V(t_0, \tau_i)\big(q_i\big(0, y(\tau_i - 0) \\ &\qquad + l_1(\tau_i - 0, \Phi(\tau_i - 0))\big) - q_i(\Phi(\tau_i - 0))\big) \end{aligned}$$

in the Banach space $\mathscr{B}_1 \times \mathscr{S}$.

Proof. Consider the operator $\mathbf{K}_1$, defined by the formula

$$\begin{aligned} \mathbf{K}_1 k_1(t_0, x_0, y_0) &= \int_{t_0}^{+\infty} U(t_0, \tau)\big(f(\tau, \Phi(\tau)) - f\big(\tau, x(\tau) + k_1(\tau, \Phi(\tau)), \\ &\qquad G\big(\tau, x(\tau) + k_1(\tau, \Phi(\tau))\big)\big)\big)\, d\tau \\ &\quad + \sum_{t_0 < \tau_i} U(t_0, \tau_i)\big(p_i(\Phi(\tau_i - 0)) \\ &\qquad - p_i\big(x(\tau_i - 0) + k_1(\tau_i - 0, \Phi(\tau_i - 0)), \\ &\qquad G\big(\tau_i - 0, x(\tau_i - 0) + k_1(\tau_i - 0, \Phi(\tau_i - 0))\big)\big)\big). \end{aligned}$$

Let us take arbitrary $k_1, k_1' \in \mathscr{B}_1$. Then

$$\begin{aligned} &|\mathbf{K}_1 k_1(t_0, x_0, y_0) - \mathbf{K}_1 k_1'(t_0, x_0, y_0)| \\ &\qquad \leq \epsilon\nu(1 + \lambda)(1 - \epsilon(1 + \lambda)\nu)^{-1} \|k_1 - k_1'\|\, |y_0 - G(t_0, x_0)|. \end{aligned}$$

Hence

$$\|\mathbf{K}_1 k_1 - \mathbf{K}_1 k_1'\| \le \lambda \|k_1 - k_1'\|.$$

For $(\mathbf{K}_1)(0)$ we obtain the estimate

$$|(\mathbf{K}_1)(0)| \le \lambda(1+\lambda)^{-1} |y_0 - G(t_0, x_0)|.$$

Consequently, we have $\|(\mathbf{K}_1)(0)\| \le \lambda(1+\lambda)^{-1}$ and

$$\|\mathbf{K}_1 k_1\| \le \|\mathbf{K}_1 k_1 - (\mathbf{K}_1)(0)\| + \|(\mathbf{K}_1)(0)\| \le \lambda\|k_1\| + \lambda(1+\lambda)^{-1}.$$

From these inequalities it follows that $\mathbf{K}_1 k_1 \in \mathscr{B}_1$ and $\mathbf{K}_1$ is a contraction. It implies that in $\mathscr{B}_1$ there is only one solution satisfying the functional equation $\mathbf{K}_1 k_1 = k_1$.

Consider, next, the operator $\mathbf{L}_1$, defined by the formula

$$\begin{aligned} &\mathbf{L}_1 l_1(t_0, x_0, y_0) \\ &\quad = \int_{-\infty}^{t_0} V(t_0, \tau)\big(g\big(\tau, 0, y(\tau) + l_1(\tau, \Phi(\tau))\big) - g(\tau, \Phi(\tau))\big)\, d\tau \\ &\qquad + \sum_{\tau_i \le t_0} V(t_0, \tau_i)\big(q_i\big(0, y(\tau_i - 0) \\ &\qquad\qquad + l_1\big(\tau_i - 0, \Phi(\tau_i - 0)\big)\big) - q_i\big(\Phi(\tau_i - 0)\big)\big). \end{aligned}$$

It is easily verified that $\|\mathbf{L}_1 l_1 - \mathbf{L}_1 l_1'\| \le \epsilon\mu\|l_1 - l_1'\|$ and $\|\mathbf{L}_1 l_1\| \le \mu(\epsilon\|l_1\| + \sup_{i,x,y} |q_i(x, y) - q_i(0, y)| + \sup_{t,x,y} |g(t, x, y) - g(t, 0, y)|)$. This implies that in $\mathscr{S}$ there is only one solution satisfying the functional equation $\mathbf{L}_1 l_1 = l_1$. Lemma 2 is proven.

Now we check that $(\zeta(\cdot), \eta(\cdot))\colon \mathbf{R} \to \mathbf{X} \times \mathbf{Y}$, where $\zeta(t) = x(t, t_0, x_0, y_0) + k_1(t, \Phi(t, t_0, x_0, y_0))$ and $\eta(t) = y(t, t_0, x_0, y_0) + l_1(t, \Phi(t, t_0, x_0, y_0))$ satisfies (5). We have for $t \ge t_0$

$$\begin{aligned} &k_1\big(t, \Phi(t, t_0, x_0, y_0)\big) \\ &\quad = \int_t^{+\infty} U(t, \tau)\big(f(\tau, \Phi(\tau)) \\ &\qquad\qquad - f\big(\tau, x(\tau) + k_1(\tau, \Phi(\tau)), G\big(\tau, x(\tau) + k_1(\tau, \Phi(\tau))\big)\big)\big)\, d\tau \\ &\qquad + \sum_{t < \tau_i} U(t, \tau_i)\big(p_i\big(\Phi(\tau_i - 0)\big) \\ &\qquad\qquad - p_i\big(x(\tau_i - 0) + k_1\big(\tau_i - 0, \Phi(\tau_i - 0)\big), \\ &\qquad\qquad\quad G\big(\tau_i - 0, x(\tau_i - 0) + k_1\big(\tau_i - 0, \Phi(\tau_i - 0)\big)\big)\big)\big) \end{aligned}$$

$$= U(t,t_0)\big(x_0 + k_1(t_0,x_0,y_0)\big) - x(t,t_0,x_0,y_0)$$
$$+ \int_{t_0}^{t} U(t,\tau) f\big(\tau, x(\tau) + k_1(\tau,\Phi(\tau)), G\big(\tau, x(\tau)$$
$$+ k_1(\tau,\Phi(\tau))\big)\big)\, d\tau$$
$$+ \sum_{t_0<\tau_i\le t} U(t,\tau_i) p_i\big(x(\tau_i-0) + k_1\big(\tau_i-0,\Phi(\tau_i-0)\big),$$
$$G\big(\tau_i-0, x(\tau_i-0) + k_1\big(\tau_i-0,\Phi(\tau_i-0)\big)\big)\big).$$

Analogously

$$l_1\big(t,\Phi(t,t_0,x_0,y_0)\big)$$
$$= V(t,t_0)\big(y_0 + l_1(t_0,x_0,y_0)\big) - y(t,t_0,x_0,y_0)$$
$$+ \int_{t_0}^{t} V(t,\tau) g\big(\tau, 0, y(\tau) + l_1(\tau,\Phi(\tau))\big)\, d\tau$$
$$+ \sum_{t_0<\tau_i\le t} V(t,\tau_i) q_i\big(0, y(\tau_i-0) + l_i\big(\tau_i-0,\Phi(\tau_i-0)\big)\big).$$

This means that $(\zeta(\cdot),\eta(\cdot))\colon \mathbf{R}\to\mathbf{X}\times\mathbf{Y}$ is a solution of (5) satisfying the initial conditions $\zeta(t_0) = x_0 + k_1(t_0,x_0,y_0)$, $\eta(t_0) = y_0 + l_1(t_0,x_0,y_0)$. The case $t \le t_0$ is treated analogously. Let $H_1(t_0,x_0,y_0) = (x_0 + k_1(t_0,x_0,y_0), y_0 + l_1(t_0,x_0,y_0))$. From the uniqueness of solutions we get for all $t\in\mathbf{R}$

$$H_1\big(t,\Phi(t,t_0,x_0,y_0)\big) = \Psi\big(t,t_0,H_1(t_0,x_0,y_0)\big).$$

The set

$$\mathscr{M}_1(\lambda) = \big(k\in\mathscr{B}_1 \,|\, |k(t,x,y) - k(t,x,y')| \le \lambda|y-y'|\big)$$

is a closed subset of the Banach space $\mathscr{B}_1$.

LEMMA 3. *There exists a unique solution of the functional equations*

$$k_2(t_0,x_0,w_0)$$
$$= \int_{t_0}^{+\infty} U(t_0,\tau)\big(f\big(\tau,x_0(\tau),G(\tau,x_0(\tau))\big)$$
$$- f\big(\tau, x_0(\tau) + k_2(\tau,x_0(\tau),\eta(\tau)), \eta(\tau)\big)\big)\, d\tau$$
$$+ \sum_{t_0<\tau_i} U(t_0,\tau_i)\big(p_i\big(x_0(\tau_i-0), G\big(\tau_i-0,x_0(\tau_i-0)\big)\big)$$
$$- p_i\big(x_0(\tau_i-0) + k_2\big(\tau_i-0,x_0(\tau_i-0),\eta(\tau_i-0)\big), \eta(\tau_i-0)\big)\big),$$

$$\eta(t) = V(t,t_0)w_0$$
$$+\int_{t_0}^{t} V(t,\tau)g\big(\tau, x_0(\tau) + k_2(\tau, x_0(\tau), \eta(\tau)), \eta(\tau)\big)\,d\tau$$
$$+\sum_{t_0<\tau_i\le t} V(t,\tau_i)q_i\big(x_0(\tau_i-0)$$
$$+k_2(\tau_i-0, x_0(\tau_i-0), \eta(\tau_i-0)), \eta(\tau_i-0)\big),$$
$$l_2(t_0,x_0,y_0)$$
$$= \int_{-\infty}^{t_0} V(t_0,\ \tau)\big(g\big(\tau, x_0(\tau) + k_2(\tau, x_0(\tau), y_0(\tau)$$
$$+l_2(\tau,\Psi(\tau))\big), y_0(\tau) + l_2(\tau,\Psi(\tau))\big) - g\big(\tau, 0, y_0(\tau)\big)\big)\,d\tau$$
$$+\sum_{\tau_i\le t_0} V(t_0,\tau_i)\big(q_i\big(x_0(\tau_i-0)$$
$$+k_2\big(\tau_i-0, x_0(\tau_i-0), y_0(\tau_i-0)$$
$$+l_2(\tau_i-0,\Psi(\tau_i-0))\big), y_0(\tau_i-0)$$
$$+l_2(\tau_i-0,\Psi(\tau_i-0))\big) - q_i\big(0, y_0(\tau_i-0)\big)\big)$$

in $\mathscr{M}_1(\lambda)\times\mathscr{S}$.

Proof. Consider the operator $\mathbf{K}_2$ defined by the formula

$$\mathbf{K}_2 k_2(t_0,x_0,w_0)$$
$$= \int_{t_0}^{+\infty} U(t_0,\tau)\big(f\big(\tau, x_0(\tau), G(\tau, x_0(\tau))\big)$$
$$-f\big(\tau, x_0(\tau) + k_2(\tau, x_0(\tau), \eta(\tau)), \eta(\tau)\big)\big)\,d\tau$$
$$+\sum_{t_0<\tau_i} U(t_0,\tau_i)\big(p_i\big(x_0(\tau_i-0), G(\tau_i-0, x_0(\tau_i-0))\big)$$
$$-p_i\big(x_0(\tau_i-0) + k_2(\tau_i-0, x_0(\tau_i-0), \eta(\tau_i-0)), \eta(\tau_i-0)\big)\big),$$

where

$$\eta(t) = V(t,t_0)w_0$$
$$+\int_{t_0}^{t} V(t,\tau)g\big(\tau, x_0(\tau) + k_2(\tau, x_0(\tau), \eta(\tau)), \eta(\tau)\big)\,d\tau$$
$$+\sum_{t_0<\tau_i\le t} V(t,\tau_i)q_i\big(x_0(\tau_i-0)$$
$$+k_2(\tau_i-0, x_0(\tau_i-0), \eta(\tau_i-0)), \eta(\tau_i-0)\big).$$

The set $\mathscr{N}_1(\lambda) = (k \in \mathscr{M}_1(\lambda) |\, \|k\| \le \lambda)$ is closed in space $\mathscr{B}_1$. Let us take arbitrary $k_2 \in \mathscr{N}_1(\lambda)$, $k_2' \in \mathscr{M}_1(\lambda)$. Then $\mathbf{K}_2 k_2 \in \mathscr{N}_1(\lambda)$ and $\|\mathbf{K}_2 k_2 - \mathbf{K}_2 k_2'\| \le \lambda \|k_2 - k_2'\|$. This implies that in $\mathscr{N}_1(\lambda)$ there is only one solution satisfying the functional equation $\mathbf{K}_2 k_2 = k_2$. In addition, the map k_2 also is the unique map in $\mathscr{M}_1(\lambda)$ satisfying the functional equation $\mathbf{K}_2 k_2 = k_2$.

Similar to Lemma 2, there is only one solution $l_2 \in \mathscr{S}$ satisfying the functional equation of Lemma 3 and the lemma is proven.

The next step is to define the map H_2 by the equality

$$H_2(t_0, x_0, y_0) = \big(x_0 + k_2\big(t_0, x_0, y_0 + l_2(t_0, x_0, y_0)\big), y_0 + l_2(t_0, x_0, y_0)\big).$$

Then the map H_2 satisfies the functional equation

$$H_2\big(t, \Psi(t, t_0, x_0, y_0)\big) = \Phi\big(t, t_0, H_2(t_0, x_0, y_0)\big)$$

for all $t \in \mathbf{R}$.

Consider the Banach space

$$\mathscr{B}_2 = \left(k \in \mathbf{PC}(\mathbf{R} \times \mathbf{X} \times \mathbf{Y}, \mathbf{X}) \,\middle|\, \sup_{t, x, y} \frac{|k(t, x, y)|}{|y + l_2(t, x, y) - G(t, x)|} < +\infty \right)$$

equipped with the norm

$$\|k\| = \sup_{t, x, y} \frac{|k(t, x, y)|}{|y + l_2(t, x, y) - G(t, x)|}.$$

LEMMA 4. *There exists a unique solution of the functional equations*

$$\begin{aligned}
k_3(t_0, x_0, y_0) &= \int_{t_0}^{+\infty} U(t_0, \tau)\Big(f\big(\tau, x_0(\tau), G(\tau, x_0(\tau))\big) \\
&\qquad - f\big(\tau, x_0(\tau) + k_3(\tau, \Psi(\tau)), G\big(\tau, x_0(\tau) + k_3(\tau, \Psi(\tau))\big)\big)\Big)\, d\tau \\
&\quad + \sum_{t_0 < \tau_i} U(t_0, \tau_i)\Big(p_i\big(x_0(\tau_i - 0), G(\tau_i - 0, x_0(\tau_i - 0))\big) \\
&\quad - p_i\big(x_0(\tau_i - 0) + k_3(\tau_i - 0, \Psi(\tau_i - 0)), \\
&\qquad G\big(\tau_i - 0, x_0(\tau_i - 0) + k_3(\tau_i - 0, \Psi(\tau_i - 0))\big)\big)\Big),
\end{aligned}$$

$$l_3(t_0, x_0, y_0)$$

$$= \int_{-\infty}^{t_0} V(t_0, \tau)\big(g\big(\tau, 0, y_0(\tau) + l_3(\tau, \Psi(\tau))\big) - g\big(\tau, 0, y_0(\tau)\big)\big)\, d\tau$$
$$+ \sum_{\tau_i \le t_0} V(t_0, \tau_i)\big(q_i\big(0, y_0(\tau_i - 0)$$
$$+ l_3(\tau_i - 0, \Psi(\tau_i - 0))\big) - q_i\big(0, y_0(\tau_i - 0)\big)\big)$$

in the Banach space $\mathscr{B}_2 \times \mathscr{S}$.

Proof. It is easy to verify that the maps defined by equalities $k_3(t_0, x_0, y_0) = 0$ and $l_3(t_0, x_0, y_0) = 0$ satisfy the functional equations. In addition, arbitrary solutions of the above functional equation satisfy the inequalities $\|k_3\| \le \lambda \|k_3\|$ and $\|l_3\| \le \epsilon\mu \|l_3\|$. This implies that the trivial solution is unique. Lemma 4 is proven.

LEMMA 5. *The identity*

$$H_1(t_0, H_2(t_0, x_0, y_0)) = (x_0, y_0)$$

holds true.

Proof. The maps $\alpha_1 : \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $\beta_1 : \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ defined by equalities

$$\begin{aligned}
&\alpha_1(t_0, x_0, y_0)\\
&\quad = k_2(t_0, x_0, y_0 + l_2(t_0, x_0, y_0))\\
&\qquad + k_1\big(t_0, x_0 + k_2\big(t_0, x_0, y_0 + l_2(t_0, x_0, y_0)\big), y_0 + l_2(t_0, x_0, y_0)\big),\\
&\qquad\quad \beta_1(t_0, x_0, y_0)\\
&\quad = l_2(t_0, x_0, y_0)\\
&\qquad + l_1\big(t_0, x_0 + k_2\big(t_0, x_0, y_0 + l_2(t_0, x_0, y_0)\big), y_0 + l_2(t_0, x_0, y_0)\big)
\end{aligned}$$

also satisfy the functional equations of Lemma 4. Besides, $\alpha_1 \in \mathscr{B}_2$ and $\beta_1 \in \mathscr{S}$. Hence $\alpha_1(t_0, x_0, y_0) = 0$ and $\beta_1(t_0, x_0, y_0) = 0$. It follows that

$$H_1(t_0, H_2(t_0, x_0, y_0)) = (x_0, y_0).$$

Lemma 5 is proven.

Consider the Banach space

$$\mathscr{B}_3 = \left(k \in \mathbf{PC}(\mathbf{R} \times \mathbf{X} \times \mathbf{Y} \times \mathbf{Y}, \mathbf{X}) \,\middle|\, \sup_{t, x, y, w} \frac{|k(t, x, y, w)|}{\max(|y - G(t, x)|, |y - w|)} < +\infty \right)$$

equipped with the norm

$$\|k\| = \sup_{t,x,y,w} \frac{|k(t,x,y,w)|}{\max(|y - G(t,x)|, |y-w|)}.$$

The set

$$\mathscr{M}_3(\lambda) = \big(k \in \mathscr{B}_3 \,|\, |k(t,x,y,w) - k(t,x,y,w')| \le \lambda |w - w'|\big)$$

is a closed subset of $\mathscr{B}_3$.

LEMMA 6. *There exists a unique solution of the functional equations*

$$\begin{aligned}
&k_4(t_0, x_0, y_0, w_0)\\
&\quad = \int_{t_0}^{+\infty} U(t_0,\tau)\big(f(\tau,\Phi(\tau))\\
&\qquad\qquad - f\big(\tau, x(\tau) + k_4(\tau,\Phi(\tau),\eta(\tau)),\eta(\tau)\big)\big)\,d\tau\\
&\qquad + \sum_{t_0<\tau_i} U(t_0,\tau_i)\big(p_i(\Phi(\tau_i - 0))\\
&\qquad\qquad -p_i\big(x(\tau_i - 0) + k_4(\tau_i - 0, \Phi(\tau_i - 0), \eta(\tau_i - 0)), \eta(\tau_i - 0)\big)\big),\\
&\eta(t) = V(t,t_0)w_0 + \int_{t_0}^{t} V(t,\tau) g\big(\tau, x(\tau) + k_4(\tau,\Phi(\tau),\eta(\tau)),\eta(\tau)\big)\,d\tau\\
&\qquad + \sum_{t_0<\tau_i\le t} V(t,\tau_i) q_i\big(x(\tau_i - 0)\\
&\qquad\qquad + k_4(\tau_i - 0, \Phi(\tau_i - 0), \eta(\tau_i - 0)), \eta(\tau_i - 0)\big),\\
&l_4(t_0, x_0, y_0)\\
&\quad = \int_{-\infty}^{t_0} V(t_0,\tau)\big(g\big(\tau, x(\tau) + k_4\big(\tau,\Phi(\tau), y(\tau)\\
&\qquad\qquad + l_4(\tau,\Phi(\tau))\big)\big), y(\tau) + l_4(\tau,\Phi(\tau))\big) - g(\tau,\Phi(\tau))\big)\,d\tau\\
&\qquad + \sum_{\tau_i\le t_0} V(t_0,\tau_i)\big(q_i\big(x(\tau_i - 0) + k_4\big(\tau_i - 0, \Phi(\tau_i - 0), y(\tau_i - 0)\\
&\qquad\qquad + l_4(\tau_i - 0, \Phi(\tau_i - 0))\big)\big), y(\tau_i - 0)\\
&\qquad\qquad + l_4(\tau_i - 0, \Phi(\tau_i - 0))\big) - q_i(\Phi(\tau_i - 0))\big)
\end{aligned}$$

in $\mathscr{M}_3(\lambda) \times \mathscr{S}$.

Proof. Consider the operator $\mathbf{K}_4$, defined by the formula

$$\begin{aligned}
\mathbf{K}_4 k_4(t_0, x_0, y_0, w_0) &\\
&= \int_{t_0}^{+\infty} U(t_0, \tau)\big(f(\tau, \Phi(\tau))\\
&\qquad - f\big(\tau, x(\tau) + k_4(\tau, \Phi(\tau), \eta(\tau)), \eta(\tau)\big)\big)\, d\tau\\
&\quad + \sum_{t_0 < \tau_i} U(t_0, \tau_i)\big(p_i(\Phi(\tau_i - 0)) - p_i\big(x(\tau_i - 0)\\
&\qquad + k_4(\tau_i - 0, \Phi(\tau_i - 0), \eta(\tau_i - 0)), \eta(\tau_i - 0)\big)\big),
\end{aligned}$$

where

$$\begin{aligned}
\eta(t) &= V(t, t_0) w_0\\
&\quad + \int_{t_0}^{t} V(t, \tau) g\big(\tau, x(\tau) + k_4(\tau, \Phi(\tau), \eta(\tau)), \eta(\tau)\big)\, d\tau\\
&\quad + \sum_{t_0 < \tau_i \le t} V(t, \tau_i) q_i\big(x(\tau_i - 0)\\
&\qquad + k_4(\tau_i - 0, \Phi(\tau_i - 0), \eta(\tau_i - 0)), \eta(\tau_i - 0)\big).
\end{aligned}$$

The set $\mathscr{N}_3(\lambda) = (k \in \mathscr{M}_3(\lambda) \mid \|k\| \le \lambda)$ is a closed subset of $\mathscr{B}_3$. Let us take arbitrary $k_4 \in \mathscr{N}_3(\lambda)$, $k_4' \in \mathscr{M}_3(\lambda)$. Then $\mathbf{K}_4 k_4 \in \mathscr{N}_3(\lambda)$ and $\|\mathbf{K}_4 k_4 - \mathbf{K}_4 k_4'\| \le \lambda \|k_4 - k_4'\|$. This implies that in $\mathscr{N}_3(\lambda)$ there is only one solution satisfying the functional equation $\mathbf{K}_4 k_4 = k_4$. In addition, the map k_4 also is the unique map in $\mathscr{M}_3(\lambda)$ satisfying the functional equation $\mathbf{K}_4 k_4 = k_4$.

As in Lemma 2 we get that there is only one solution $l_4 \in \mathscr{S}$ satisfying the functional equation. Note that $k_4(t_0, x_0, y_0, y_0) = 0$ and $l_4(t_0, x_0, y_0) = 0$. Lemma 6 is proven.

LEMMA 7. *The following identity*

$$H_2(t_0, H_1(t_0, x_0, y_0)) = (x_0, y_0)$$

holds true.

Proof. The maps $\alpha_2 : \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \times \mathbf{Y} \to \mathbf{X}$ and $\beta_2 : \mathbf{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ defined by the equalities

$$\alpha_2(t_0, x_0, y_0, w_0) = k_1(t_0, x_0, y_0) + k_2(t_0, x_0 + k_1(t_0, x_0, y_0), w_0),$$

$$\begin{aligned}
\beta_2(t_0, x_0, y_0)&\\
&= l_1(t_0, x_0, y_0) + l_2(t_0, x_0 + k_1(t_0, x_0, y_0), y_0 + l_1(t_0, x_0, y_0))
\end{aligned}$$

also satisfy the functional equations of Lemma 6. Besides, $\beta_2 \in \mathscr{S}$ and $\alpha_s \in \mathscr{M}_3(\lambda)$. Hence $\alpha_2(t_0, x_0, y_0, y_0) = 0$ and $\beta_2(t_0, x_0, y_0, y_0) = 0$. We get $H_2(t_0, H_1(t_0, x_0, y_0)) = (x_0, y_0)$. Lemma 7 is proven.

Taking into account Lemmas 5 and 7, we get that $H_1(t_0, \cdot)$ is a homeomorphism establishing the strong global dynamical equivalence of systems (1) and (5). It is easy to verify that if the system (1) of differential equations is autonomous and without impulse effect, then the maps G, H_1, and H_2 are independent of $t_0 \in \mathbf{R}$. Let us note that in our case $e(x, y) = a + \epsilon|y|$, where a is some positive constant. Thus, the proof of the theorem is complete.

6. CONCLUSION

By reversing time t one can prove an analogous theorem for the system of differential equations with impulse effect at fixed moments

$$\begin{cases} dx/dt = A(t)x + f(t, x, z), \\ dz/dt = C(t)z + h(t, x, z), \\ \Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) \\ \qquad\qquad = D_i x(\tau_i - 0) + p_i\big(x(\tau_i - 0), z(\tau_i - 0)\big), \\ \Delta z|_{t=\tau_i} = z(\tau_i + 0) - z(\tau_i - 0) \\ \qquad\qquad = F_i z(\tau_i - 0) + r_i\big(x(\tau_i - 0), z(\tau_i - 0)\big), \end{cases} \tag{10}$$

where the Cauchy evolution operators $U(t, \tau)$, $W(t, \tau)$ of the corresponding linear system satisfy inequalities

$$\nu = \max\Bigg(\sup_t \int_t^{+\infty} |W(t, \tau)|\,|U(\tau, t)|\, d\tau + \sup_t \sum_{t<\tau_i} |W(t, \tau_i)|\,|U(\tau_i - 0, t)|,$$

$$\sup_t \int_{-\infty}^t |W(\tau, t)|\,|U(t, \tau)|\, d\tau + \sup_t \sum_{\tau_i \le t} |W(\tau_i - 0, t)|\,|U(t, \tau_i)| \Bigg) < +\infty,$$

$$\mu = \sup_t \left(\int_t^{+\infty} |W(t, \tau)|\, d\tau + \sum_{t<\tau_i} |W(t, \tau_i)| \right) < +\infty.$$

Then by relating Theorem 2 to Eq. (1) with the analogous theorem being applied to the first part of the newly obtained system one can prove the reduction theorem.

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